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by

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1 Introduction

In this paper we investigate two approaches to the aggregation of the Luenberger productivity indicator. Our first approach imposes allocative efficiency of every observed input-output vector with respect to the technologies in every time period. Our second approach only imposes allocative efficiency of observed input-output vectors with respect to their contemporaneous technologies. This approach utilizes the superlative index number approach pioneered by Diewert (1976) and applied to directional distance functions by Balk (1998).

Our two approaches may be contrasted with a paper by Blackorby and Russell (1999). In that paper they posed a general question of whether efficiency indexes of individual firms would exactly aggregate to the industry level. They made no assumptions about the behavior of the firms (or the industry). By insisting that aggregation be exact for arbitrary values of the quantity variables they were able to establish rather negative results about the possibility of exact aggregation.

In the present paper, we specialize the aggregation question to that of aggregating Luenberger productivity indicators. In this context, our main goal is to show that the possibility of exact aggregation can be improved albeit at the cost of making assumptions about the behavior of the firms and the industry. However, if such aggregation is possible then researchers who have constructed productivity indicators for each firm in an industry will also be able to readily calculate a productivity indicator for the industry.

2 The Luenberger Productivity Indicator

The Luenberger productivity indicator is defined by differences in values of the directional distance function. Thus, we introduce such a distance function defined on the technology set, T, where

$$T = \left\{(x, y) : \text{input } x \in R_+^N \text{ can produce output } y \in R_+^M\right\}$$

and

$$\vec{D}(x, y; g_x, g_y) = \sup_\beta \{\beta : (x - \beta g_x, y + \beta g_y) \in T\}$$

is the directional distance function defined on T for the direction vector (g_x, g_y). Defining this function for time period t we get:

$$\vec{D}^t(x^t, y^t; g_x, g_y) = \sup_\beta \left\{\beta : (x^t - \beta g_x, y^t + \beta g_y) \in T^t\right\}$$

Chambers (1996) and Chambers, Färe, and Grosskopf (1996) define the Luenberger productivity indicator as

$$\begin{aligned} &\mathsf{L}(x^t, y^t, x^{t+1}, y^{t+1}; g_x, g_y) \\ = \; &\frac{1}{2}\left\{ \begin{array}{l} \left[\vec{D}^{t+1}(x^t, y^t; g_x, g_y) - \vec{D}^{t+1}(x^{t+1}, y^{t+1}; g_x, g_y)\right] \\ + \left[\vec{D}^t(x^t, y^t; g_x, g_y) - \vec{D}^t(x^{t+1}, y^{t+1}; g_x, g_y)\right] \end{array} \right\}. \end{aligned} \tag{1}$$

This indicator is the arithmetic average of the productivity change measured by the technology at time $t + 1$ (the first two terms) and the productivity change measured by the technology at time t (the last two terms).

We want to study the possibility of aggregating firm productivity indicators to form the industry indicator.[1] However, the aggregation problems that we pose below require a heavy dose of additional notation. Thus, in the remainder of this paper, we shall suppress the direction vector (g_x, g_y). For example, we replace $\vec{D}^t(x^t, y^t; g_x, g_y)$ with $\vec{D}^t(x^t, y^t)$ and we replace $\mathsf{L}(x^t, y^t, x^{t+1}, y^{t+1}; g_x, g_y)$ with $\mathsf{L}(x^t, y^t, x^{t+1}, y^{t+1})$.

There are K firms in the industry numbered $k = 1, ..., K$. Their technologies at time t are represented by

$$T^{k,t} = \left\{\left(x^{k,t}, y^{k,t}\right) : x^{k,t} \in R_+^N \text{ can produce } y^{k,t} \in R_+^M\right\}, k = 1, ..., K.$$

We now define the industry (aggregate) technology, denoted by $T^{0,t}$, and the industry input and output vectors by

$$T^{0,t} = \sum_{k=1}^K T^{k,t}, x^{0,t} = \sum_{k=1}^K x^{k,t} \text{ and } y^{0,t} = \sum_{k=1}^K y^{k,t}, \tag{2}$$

(see Koopmans (1957)). For the industry and for each firm the directional distance function is

$$\vec{D}^{k,t}(x^{k,t}, y^{k,t}) = \sup_\beta \left\{\beta : (x^{k,t} - \beta g_x, y^{k,t} + \beta g_y) \in T^{k,t}\right\}, k = 0, 1, ..., K.$$

[1] Or, stated differently, we want to study the disaggregation of the industry indicator into firm indicators.

3 The Aggregation Problem

The aggregation problem can now be stated as follows. Under what condition(s) is

$$\begin{aligned} & \mathsf{L}^0(x^{0,t}, y^{0,t}, x^{0,t+1}, y^{0,t+1}) \\ = & \sum_{k=1}^{K} \mathsf{L}^k \left(x^{k,t}, y^{k,t}, x^{k,t+1}, y^{k,t+1} \right), \end{aligned} \tag{3}$$

where

$$\begin{aligned} & \mathsf{L}^k(x^{k,t}, y^{k,t}, x^{k,t+1}, y^{k,t+1}) \\ = & \frac{1}{2} \left\{ \begin{array}{c} \left[\vec{D}^{k,t+1}(x^{k,t}, y^{k,t}) - \vec{D}^{k,t+1}(x^{k,t+1}, y^{k,t+1}) \right] \\ + \left[\vec{D}^{k,t}(x^{k,t}, y^{k,t}) - \vec{D}^{k,t}(x^{k,t+1}, y^{k,t+1}) \right] \end{array} \right\}, \end{aligned} \tag{4}$$

$k = 0, 1, ..., K$?

Rewriting (3) using (4) and (2) we get

$$\begin{aligned} = & \frac{1}{2} \left\{ \begin{array}{c} \left[\vec{D}^{0,t+1} \left(\sum_{k=1}^{K} x^{k,t}, \sum_{k=1}^{K} y^{k,t} \right) - \vec{D}^{0,t+1} \left(\sum_{k=1}^{K} x^{k,t+1}, \sum_{k=1}^{K} y^{k,t+1} \right) \right] \\ + \left[\vec{D}^{0,t} \left(\sum_{k=1}^{K} x^{k,t}, \sum_{k=1}^{K} y^{k,t} \right) - \vec{D}^{0,t} \left(\sum_{k=1}^{K} x^{k,t+1}, \sum_{k=1}^{K} y^{k,t+1} \right) \right] \end{array} \right\} \\ = & \sum_{k=1}^{K} \left[\frac{1}{2} \left\{ \begin{array}{c} \left[\vec{D}^{k,t+1}(x^{k,t}, y^{k,t}) - \vec{D}^{k,t+1}(x^{k,t+1}, y^{k,t+1}) \right] \\ + \left[\vec{D}^{k,t}(x^{k,t}, y^{k,t}) - \vec{D}^{k,t}(x^{k,t+1}, y^{k,t+1}) \right] \end{array} \right\} \right]. \end{aligned} \tag{5}$$

A sufficient condition for (5) is that each of the four individual terms on each side are equal. This will be true if

$$\vec{D}^t \left(\sum_{k=1}^{K} x^k, \sum_{k=1}^{K} y^k \right) = \sum_{k=1}^{K} \vec{D}^{k,t}(x^k, y^k), \text{ for all } (x^k, y^k) \text{ and for all } t. \tag{6}$$

This is a special case of the type of aggregation problem posed by Blackorby and Russell (1999). It is a Pexider-Sincov equation; the solution is given by Aczél (1966, page 302). The resulting restrictions on the directional distance functions are very strong. In particular, (6) implies that

$$\vec{D}^{k,t}(x^k, y^k) = \varepsilon^{k,t} \left(\sum_{n=1}^{N} a_n x_n^k + \sum_{m=1}^{M} b_m y_m^k \right), k = 1, ..., K,$$

and

$$\vec{D}^{0,t} \left(x^0, y^0 \right) = \varepsilon \left(\sum_{n=1}^{N} a_n x_n^0 + \sum_{m=1}^{M} b_m y_m^0 \right).$$

(See Theorem 1 in Blackorby and Russell (1999)).

However, we will show that firm-level Luenberger productivity indicators can be aggregated under alternative conditions. Denote two time periods by a and b (where a and b can refer to the same time period or to two adjacent time periods.) The requirement that each of the four individual terms on each side of (5) are equal can be written more succinctly as

$$\vec{D}^{0,a}\left(\sum_{k=1}^{K} x^{k,b}, \sum_{k=1}^{K} y^{k,b}\right) = \sum_{k=1}^{K} \vec{D}^{k,a}(x^{k,b}, y^{k,b}), a = t, t+1, b = t, t+1. \qquad (7)$$

Next, define the profit functions, for the industry and for each firm,

$$\begin{aligned} \Pi^{k,a}(p^a, w^a) &= \max_{x^{k,a}, y^{k,a}} \left\{p^a y^{k,a} - w^a x^{k,a} : (x^{k,a}, y^{k,a}) \in T^{k,a}\right\} \\ &= \max_{x^{k,a}, y^{k,a}} \left\{p^a y^{k,a} - w^a x^{k,a} : \vec{D}^{k,a}(x^{k,a}, y^{k,a}) \geq 0\right\} \end{aligned}$$

for $a = t, t+1,\ k = 0, 1, ..., K$.

The profit function can be used to define Nerlovian profit efficiency for the industry and for each firm as

$$PE^{k,a}\left(w^a, x^b, p^a, y^b\right) = \frac{\Pi^{k,a}(p^a, w^a) - (p^a y^{k,b} - w^a x^{k,b})}{p^a g_y + w^a g_x},$$

for $a = t, t+1, b = t, t+1$ and $k = 0, 1, ..., K$. (See Chambers, Chung, and Färe (1998) who define profit efficiency when $a = b$). The directional distance function is used to define a measure of technical efficiency for the industry and for each firm

$$TE^{k,a}(x^{k,b}, y^{k,b}) = \vec{D}^{k,a}(x^{k,b}, y^{k,b}),\ a = t, t+1, b = t, t+1,$$

$k = 0, 1, ..., K$. Allocative efficiency, AE, is then given by the "residual" of the following decomposition

$$\begin{aligned} & PE^{k,a}\left(w^a, x^{k,b}, p^a, y^{k,b}\right) \\ = \ & TE^{k,a}(x^{k,b}, y^{k,b}) + AE^{k,a}\left(w^a, x^{k,b}, p^a, y^{k,b}\right), \end{aligned} \qquad (8)$$

$a = t, t+1, b = t, t+1, k = 0, 1, ..., K$.

To illustrate (8) consider the following one-input one-output diagram.

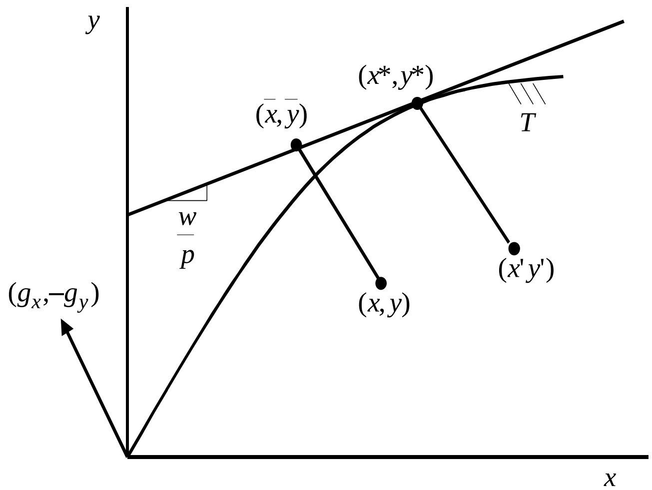

Figure 1

In this example, (x^*, y^*) is profit maximizing at prices (p, w) and $PE(w, x^*, p, y^*) = TE(x^*, y^*) = AE(w, x^*, p, y^*) = 0$. The point (x', y') is allocatively efficient but not technically efficient so that $TE(x', y') > 0$ and $AE(w, x', p, y') = 0$. The point $(\bar{x}, \bar{y})$ is technically efficient but not allocatively efficient, i.e., $TE(\bar{x}, \bar{y}) = 0$ and $AE(w, \bar{x}, p, \bar{y}) > 0$. Finally, (x, y) is neither technically nor allocatively efficient and $TE(x, y) > 0$ and $AE(w, x, p, y) > 0$.

We also note, in passing, that

$$\begin{aligned} \Pi(p, w) &= py^* - wx^* \\ &= p(y' + \vec{D}(x', y')g_y) - w(x' - \vec{D}(x', y')g_x). \end{aligned} \tag{9}$$

A standard result, nicely characterized in Koopmans (1957), is that

$$\Pi^{0,a}(p^a, w^a) = \sum_{k=1}^{K} \Pi^{k,a}(p^a, w^a), a = t, t+1. \tag{10}$$

Using (10) it is clear that

$$\begin{aligned} PE^{0,a}\left(w^a, x^{0,b}, p^a, y^{0,b}\right) &= \frac{\Pi^{0,a}(p^a, w^a) - (p^a y^{0,b} - w^a x^{0,b})}{p^a g_y + w^a g_x} \\ &= \frac{\sum_{k=1}^{K} \Pi^{k,a}(p^a, w^a) - (p^a y^{0,b} - w^a x^{0,b})}{p^a g_y + w^a g_x} \\ &= \sum_{k=1}^{K} \frac{\Pi^{k,a}(p^a, w^a) - (p^a y^{k,b} - w^a x^{k,b})}{p^a g_y + w^a g_x} \end{aligned}$$

$$= \sum_{k=1}^{K} PE^{k,a}\left(w^a, x^{k,b}, p^a, y^{k,b}\right). \tag{11}$$

$a = t, t+1, b = t, t+1$. Then, using (8) and (11) we get

$$\begin{aligned} & TE^{0,a}(x^{0,b}, y^{0,b}) + AE^{0,a}\left(w^a, x^{0,b}, p^a, y^{0,b}\right) \\ = & \sum_{k=1}^{K} TE^{k,a}(x^{k,b}, y^{k,b}) + \sum_{k=1}^{K} AE^{k,a}\left(w^a, x^{k,b}, p^a, y^{k,b}\right) \end{aligned} \tag{12}$$

$a = t, t+1, b = t, t+1$.

We can now state the result of this section: If

$$AE^{0,a}\left(w^a, x^{0,b}, p^a, y^{0,b}\right) = \sum_{k=1}^{K} AE^{k,a}\left(w^a, x^{k,b}, p^a, y^{k,b}\right), a = t, t+1, b = t, t+1 \tag{13}$$

then (7) holds and thus

$$\mathsf{L}^0(x^{0,t}, y^{0,t}, x^{0,t+1}, y^{0,t+1}) = \sum_{k=1}^{K} \mathsf{L}^k\left(x^{k,t}, y^{k,t}, x^{k,t+1}, y^{k,t+1}\right)$$

"Proof" Given (12) and (13) we get

$$TE^{0,a}(x^{0,b}, y^{0,b}) = \sum_{k=1}^{K} TE^{k,a}(x^{k,b}, y^{k,b}), a = t, t+1, b = t, t+1$$

i.e.,

$$\vec{D}^{0,a}\left(x^{0,b}, y^{0,b}\right) = \sum_{k=1}^{K} \vec{D}^{k,a}(x^{k,b}, y^{k,b}), a = t, t+1, b = t, t+1.$$

This is just (7).

How likely is it that (13) will hold? A sufficient condition is that

$$AE^{0,a}\left(w^a, x^{0,b}, p^a, y^{0,b}\right) = AE^{k,a}\left(w^a, x^{k,b}, p^a, y^{k,b}\right) = 0, \tag{14}$$

$a = t, t+1, b = t, t+1, k = 1, ..., K$ i.e., each observation is allocatively efficient in both time periods. For a study that covers several time periods this implies that each observation $(x^{k,t}, y^{k,t})$ is allocatively efficient with respect to the technologies in all time periods. For example, for four time periods, this condition implies the following sort of diagram.

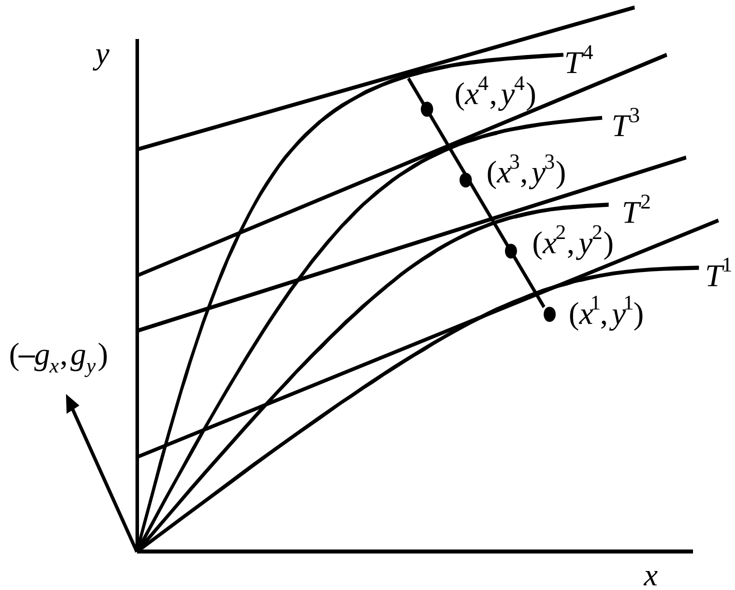

Figure 2

As the technology shifts over time and prices change all of the profit-maximizing input-output vectors and all of the observed input-output vectors must lie along the same direction given by $(-g_x, g_y)$. This is quite a strong condition. In the next section we find alternative conditions that are more palatable.

4 The Superlative Index Number Approach

In this section, we will make the assumption that in each time period, t, the observed choices, $(x^{0,t}, y^{0,t})$and $(x^{k,t}, y^{k,t})$ are allocatively efficient relative to the technologies, $T^{0,t}$ and $T^{k,t}, k = 1, ..., K$. This implies that

$$\Pi^{k,t}\left(p^t, w^t\right) = p^t\left(y^{k,t} + \vec{D}^{k,t}(x^{k,t}, y^{k,t})g_y\right) - w^t\left(x^{k,t} - \vec{D}^{k,t}(x^{k,t}, y^{k,t})g_x\right), \tag{15}$$

$k = 0, 1, ..., K$. (Recall Figure 1 and the discussion that led to (9)). The economic interpretation of (15) is this: At prices (p^t, w^t) the output-input vector

$$\left(y^{0,t} + \vec{D}^{0,t}(x^{0,t}, y^{0,t})g_y, x^{0,t} - \vec{D}^{0,t}(x^{0,t}, y^{0,t})g_x\right)$$

is profit-maximizing for the industry and the output-input vector

$$\left(y^{k,t} + \vec{D}^{k,t}(x^{k,t}, y^{k,t})g_y, x^{k,t} - \vec{D}^{k,t}(x^{k,t}, y^{k,t})g_x\right)$$

is profit-maximizing for firm $k, k=1,...,K$. In addition, since these profit-maximizing choices are homogeneous of degree zero in prices then they are also profit-maximizing at the normalized price vector, $(\hat{p}^t, \hat{w}^t)$, that is defined by

$$(\hat{p}^t, \hat{w}^t) = \left(\frac{p^t}{p^t g_y + w^t g_x}, \frac{w^t}{p^t g_y + w^t g_x} \right).$$

The second assumption that we use is that the industry and firm technologies can be modelled by directional distance functions, $\vec{D}^{k,t}(x^{k,t}, y^{k,t}), k = 0, 1, ..., K$, that are all quadratic functions with second-order coefficients that are time-invariant. We make this assumption in the same spirit as Diewert (1976) who introduced the notion of superlative index numbers. Thus, for the industry and for each firm,

$$\begin{aligned} \vec{D}^{k,t}(x^{k,t}, y^{k,t}) &= a_0^{k,t} + a^{k,t} x^{k,t} + \frac{1}{2} \left(x^{k,t}\right)' A^k x^{k,t} \\ &\quad + b^{k,t} y^{k,t} + \frac{1}{2} \left(y^{k,t}\right)' B^k y^{k,t} + \left(x^{k,t}\right)' C^k y^{k,t}, \end{aligned} \tag{16}$$

$k = 0, 1, ..., K$. We are now in a position to apply Theorem 7.2 in Balk (1998, page 175) that states the following. For the industry and for each firm, if (15) and (16) hold then the Luenberger productivity indicator may be calculated by

$$\begin{aligned} & \mathsf{L}^k(x^{k,t}, y^{k,t}, x^{k,t+1}, y^{k,t+1}) \\ = \; & \frac{1}{2} \left(\hat{p}^t + \hat{p}^{t+1}\right) \left(y^{k,t+1} - y^{k,t}\right) - \frac{1}{2} \left(\hat{w}^t + \hat{w}^{t+1}\right) \left(x^{k,t+1} - x^{k,t}\right). \end{aligned} \tag{17}$$

$k = 0, 1, ..., K$.

Our main result in this section is this: Given the assumptions of allocative efficiency, (15), and quadratic form with time-invariant second-order coefficients, (15), the Luenberger productivity indicator for the industry is the simple sum of the Luenberger productivity indicators for the individual firms.

"Proof":

$$\begin{aligned} & \mathsf{L}^0(x^{0,t}, y^{0,t}, x^{0,t+1}, y^{0,t+1}) \\ = \; & \frac{1}{2} \left(\hat{p}^t + \hat{p}^{t+1}\right) \left(\sum_{k=1}^K y^{k,t+1} - \sum_{k=1}^K y^{k,t} \right) - \frac{1}{2} \left(\hat{w}^t + \hat{w}^{t+1}\right) \left(\sum_{k=1}^K x^{k,t+1} - \sum_{k=1}^K x^{k,t} \right) \\ = \; & \sum_{k=1}^K \left[\frac{1}{2} \left(\left(\hat{p}^t + \hat{p}^{t+1}\right) \left(y^{k,t+1} - y^{k,t}\right) \right) - \frac{1}{2} \left(\left(\hat{w}^t + \hat{w}^{t+1}\right) \left(x^{k,t+1} - x^{k,t}\right) \right) \right] \\ = \; & \sum_{k=1}^K \mathsf{L}^k(x^{k,t}, y^{k,t}, x^{k,t+1}, y^{k,t+1}). \end{aligned}$$

Remark: This aggregation result is, of course, due to the linearity of the Luenberger productivity indicator. It is also important that all firms face the same prices and maximize profit at their technical efficiency adjusted quantity vectors and it is important that they face the same normalized prices, i.e., that the same director vector applies to all firms.

5 Is Allocative Efficiency Realistic?

We have made extensive use of the assumption that all firms and the industry are allocatively efficient. As useful as this assumption is[2], how realistic is it? Of course, it is unrealistic to assume that firms are exactly allocatively efficient but can a case be made that firms are approximately so? Put a different way, is there an argument that errors made by firms in achieving allocative efficiency do not lead to large profit losses? If they do not then a case can be made that the size of the measure of the allocative inefficiency may be "small".

In this section we drop all notation that indexes either time or firms. To focus on allocative efficiency we also assume that each (x, y) has been adjusted for technical efficiency, i.e., replaced by $(\bar{x}, \bar{y}) = \left(x - \vec{D}(x, y)g_x, y + \vec{D}(x, y)g_y\right)$.

Allocative efficiency implies that $(\bar{x}, \bar{y})$ maximizes (industry) profit. Suppose that it does not. Then let (x^*, y^*) be the profit-maximizing choice. We now follow the line of reasoning similar to that presented in Akerlof and Yellen (1985) who argued that in economic models, first-order deviations of economic variables from their optimal values result in only second-order differences in the value of the optimal value function.

The profit function is derived as:

$$\begin{aligned}\Pi(p, w) &= \max\left\{py - wx : \vec{D}(x, y) = 0\right\} \\ &= \max\left\{p\left(y + \vec{D}(x, y)g_y\right) - w\left(x - \vec{D}(x, y)g_x\right)\right\} \\ &= \max\left\{f(x, y)\right\},\end{aligned}$$

where $f(x, y) = p\left(y + \vec{D}(x, y)g_y\right) - w\left(x - \vec{D}(x, y)g_x\right)$. Differentiating $f(x, y)$ twice, we note that

$$\nabla_{xx}f(x, y) = \nabla_{xx}\vec{D}(x, y)\left(pg_y + wg_x\right) \tag{18}$$

$$\nabla_{yy}f(x, y) = \nabla_{yy}\vec{D}(x, y)\left(pg_y + wg_x\right) \tag{19}$$

$$\nabla_{xy}f(x, y) = \nabla_{xy}\vec{D}(x, y)\left(pg_y + wg_x\right) \tag{20}$$

The allocative efficiency measure can be written as

$$\begin{aligned}AE\left(w, \bar{x}, p, \bar{y}\right) &= PE\left(w, \bar{x}, p, \bar{y}\right) \\ &= \frac{f(x^*, y^*) - f(\bar{x}, \bar{y})}{pg_y + wg_x},\end{aligned} \tag{21}$$

[2]In a related paper, Rolf Färe, Shawna Grosskopf, and Valentin Zelenyuk (2001) also make use of the assumption of allocative efficiency.

since $(\bar{x}, \bar{y})$ is technically efficient. The numerator in (21) is the loss of profit that results from choosing $(\bar{x}, \bar{y})$ instead of (x^*, y^*).

A second-order Taylor series approximation of f around the point (x^*, y^*) evaluated at the point $(\bar{x}, \bar{y})$ provides the following expression:

$$\begin{aligned} f(\bar{x}, \bar{y}) &= f(x^*, y^*) + \nabla_x f(x^*, y^*)\,(\bar{x} - x^*) + \nabla_y f(x^*, y^*)\,(\bar{y} - y^*) \\ &\quad + \frac{1}{2}\,(\bar{x} - x^*)'\,\nabla_{xx} f(\tilde{x}, \tilde{y})\,(\bar{x} - x^*) + \frac{1}{2}\,(\bar{y} - y^*)'\,\nabla_{yy} f(\tilde{x}, \tilde{y})\,(\bar{y} - y^*) \\ &\quad + (\bar{x} - x^*)'\,\nabla_{xy} f(\tilde{x}, \tilde{y})\,(\bar{y} - y^*)\,. \end{aligned} \tag{22}$$

where $(\tilde{x}, \tilde{y}) = \alpha\,(\bar{x}, \bar{y}) + (1 - \alpha)\,(x^*, y^*)\,, 0 \leq \alpha \leq 1$. However, the first-order conditions for profit maximization force the second and third terms on the right hand side of (22) to be zero. Rearranging, we get

$$\begin{aligned} & f(x^*, y^*) - f(\bar{x}, \bar{y}) = \\ & -\frac{1}{2}\Delta x' \nabla_{xx} f(\tilde{x}, \tilde{y}) \Delta x - \frac{1}{2}\Delta y' \nabla_{yy} f(\tilde{x}, \tilde{y}) \Delta y - \Delta x' \nabla_{xy} f(\tilde{x}, \tilde{y}) \Delta y \end{aligned} \tag{23}$$

where $\Delta x = (\bar{x} - x^*)$ and $\Delta y = (\bar{y} - y^*)$. Thus the profit loss due to nonmaximizing behavior depends only on second order terms in the above Taylor series expansion.

Substituting (18) - (20) and (23) into (21) we get

$$\begin{aligned} & AE\,(w, \bar{x}, p, \bar{y}) = \\ & -\frac{1}{2}\Delta x' \nabla_{xx} \vec{D}(\tilde{x}, \tilde{y}) \Delta x - \frac{1}{2}\Delta y' \nabla_{yy} \vec{D}(\tilde{x}, \tilde{y}) \Delta y - \Delta x' \nabla_{xy} \vec{D}(\tilde{x}, \tilde{y}) \Delta y. \end{aligned}$$

Thus, the measure of allocative efficiency will be "small" if either the deviations Δx and Δy are small or if the second order derivatives of the directional distance function are small.

6 Conclusion

The main purpose of our study has been to show that aggregation of Luenberger productivity indicators is possible under assumptions of allocative efficiency. With our first approach, aggregation is possible if both observed quantity vectors are allocatively efficient with respect to both of the time-adjacent technologies. With the superlative index number approach, aggregation is possible if each observed quantity vector is allocatively efficient with respect to the current technology and if the directional distance function has a quadratic functional form with time-independent second order coefficients. We conclude that the superlative index number approach is the more promising of the two.

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